Excitations for lattice ferromagnetic classical spin systems at high temperature: Noneven single-spin distributions

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We consider general d-dimensional lattice ferromagnetic spin systems with nearest neighbor interactions in the high temperature region ($\beta \ll 1$). Each model is characterized by a single-site *a priori* spin probability distribution taken to be noneven. We state our results in terms of the parameter $\alpha = (\langle \bar{s}^4 \rangle - 3\langle \bar{s}^2 \rangle^2 - \langle \bar{s}^3 \rangle^2 \langle \bar{s}^2 \rangle^{-1})/(\langle \bar{s}^4 \rangle - \langle \bar{s}^2 \rangle^2 - \langle \bar{s}^3 \rangle^2 \langle \bar{s}^2 \rangle^{-1})$, where $\bar{s} = s - \langle s \rangle$, and $\langle s^k \rangle$ denotes the kth moment of the single-site distribution. Associated with the model is a lattice quantum field theory which is known to contain a particle of mass $m \sim \ln \beta$. Assuming $\langle \bar{s}^3 \rangle \neq 0$ we show that for $\alpha > 0$, β small, there exists a bound state with mass below the two-particle threshold $2m$. For $\alpha < 0$ bound states do not exist. These results are obtained using a Bethe-Salpeter (BS) equation in the ladder approximation in conjunction with a representation for the inverse of the two-point function designed to analyze the spectrum below but close to $2m$.

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I. INTRODUCTION AND RESULTS

In this work we obtain properties of general d-dimensional lattice ferromagnetic classical spin systems with nearest neighbor interactions in the high temperature region. Each such system is characterized by a single-site *a priori* spin probability distribution (SSD). Associated with these systems is a lattice quantum field theory with Hamiltonian energy and field momentum operators living on a $(d-1)$-dimensional sublattice. The Hamiltonian is minus the logarithm of the transfer matrix (see Refs. [1,2]). The properties are uncovered by a detailed study of the interaction of the particles of this underlying field theory. The idea of studying these systems via the transfer matrix is not new, but up to now it has only been established that the low-lying energy-momentum (e-m) spectrum consists of a particle with isolated dispersion curve. These results imply exponential decay of correlation functions (CF) and the Ornstein-Zernike behavior of the two-point CF [3,4]. Our results go beyond this giving information on the spectrum up to the two-particle threshold, and have consequences for the decay of CF's.

Previously (see Ref. [5]) we considered even SSD and found that the sign of the parameter

$$\alpha' \equiv \langle s^4 \rangle - 3\langle s^2 \rangle^2,$$

where $\langle s^k \rangle$ denotes the kth moment of the single-spin probability distribution, and determines the presence or absence of bound states below the two-particle threshold. We showed that if $\alpha' > 0$ the dominant interaction (which is local) is attractive and a bound state exists, such that there is an energy-momentum spectrum between the one-particle spectrum and below the two-particle threshold; for $\alpha' > 0$ the spectrum is absent. The results were obtained by establishing a spectral representation for a vacuum subtracted four-point function, and analyzing the CF utilizing a lattice Bethe-Salpeter equation in the ladder approximation.

Here we consider the case of a noneven SSD. One might expect that the relevant parameter is α' but using $\bar{s} \equiv s - \langle s \rangle$ in place of s. However, this is not so. Our basic result is that, letting

$$\alpha \equiv \frac{\langle \bar{s}^4 \rangle - 3\langle \bar{s}^2 \rangle^2 - \langle \bar{s}^3 \rangle^2 / \langle \bar{s}^2 \rangle}{\langle \bar{s}^4 \rangle - \langle \bar{s}^3 \rangle^2 / \langle \bar{s}^2 \rangle}, \tag{1.1}$$

if $\alpha > 0$, and β is sufficiently small then a bound state exists. For $\alpha < 0$ there is no spectrum between the one-particle mass and the two-particle threshold. These spectral results are obtained assuming that $\langle \bar{s}^3 \rangle \neq 0$ using a ladder approximation of a lattice BS equation in conjunction with a representation for the inverse the two-point function tailored to determine the spectrum close to but below the two-particle spectrum. We assume $\langle \bar{s}^3 \rangle \neq 0$ to exclude, in the lowest moment, a noneven SSD obtained by a constant shift of an even SSD. These approximations have proved to be reliable where a complete mathematical treatment can be given (see Refs. [6–8]).

We give a more precise description of the class of models we treat. We let $s(x) \in R$, and $x = (x_0, \vec{x}) \in \Lambda \subset Z^d$ denote the spin variable at the site x of the finite lattice Λ. For the generating function $Z_\Lambda(J)$ we take $Z_\Lambda(J) = \int e^{(J,s)} e^{S(s)} d\mu(s)$ and $(J,s) = \Sigma_x J(x) s(x)$, and the interacting action $S(s)$ is $S(s) = \beta \Sigma' s(x) s(y)$, where Σ' denotes the sum over the unordered set of nearest neighbor sites $\{x,y\}$. $d\mu(s) = \Pi_x e^{-V(s(x))} ds(x)$ and we only consider the case of noneven SSD, i.e., $V(s) \neq V(-s)$. $V(s)$ is bounded from below and increases at infinity at least quadratically. Expectations of the probability measure $\exp[S(s)] d\mu(s)$/normalization are denoted by $\langle \cdot \rangle_\Lambda$. Truncated CF's are given by local derivatives with respect to J's of $\ln Z_\Lambda(J)$ at $J=0$. By the polymer expansion (see Ref. [4]) the thermodynamic limit ($\Lambda \to Z^d$) of the CF's exist if β is sufficiently small. The limiting CF's are denoted by $\langle \cdot \rangle$ and are translation invariant. The truncated CF's have exponential tree decay.

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We now motivate the method for obtaining these spectral results in the language of the associated imaginary discrete time lattice quantum field theory (QFT). The QFT is constructed in the standard way (see Refs. [1,2]). Write $x=(x_0,\vec{x})\in Z^d$ and take the x_0 direction as the discrete imaginary time. The construction furnishes us with the quantum mechanical Hilbert space $\mathcal{H}$, the commuting self-adjoint energy $H\geqslant 0$, and the momentum operators $\vec{P}$, the vacuum vector Ω, and the time zero local-field operators which we denote by $\hat{s}(\vec{x})$, and $\vec{x}\in Z^{d-1}$. The CF of the spin variable $s(x)$, $x=(x_0,\vec{x})\in Z^d$, is related to the vacuum expectations of the QFT by the Feynman-Kac (FK) formula, and the decay rates of the CF are used to obtain spectral information. The relation of the Hilbert space objects to the CF's is given by the FK formula, i.e., setting $\hat{s}(0)=\hat{s}$, and $x_k=(t_k\vec{x}_k)$, with

$$t_1\leqslant t_2\leqslant\ldots\leqslant t_n,$$

$$\begin{aligned}&(\Omega,\hat{s}e^{-H(t_2-t_1)}e^{i\vec{P}\cdot(\vec{x}_2-\vec{x}_1)}\hat{s}e^{-H(t_3-t_2)}\\&\qquad\times e^{i\vec{P}\cdot(\vec{x}_3-x_2)}\ldots e^{-H(t_n-t_{n-1})}e^{i\vec{P}\cdot(\vec{x}_n-\vec{x}_{n-1})}\hat{s})\\&\quad=\langle s(x_1)s(x_2)\cdots S(x_n)\rangle.\end{aligned}$$

One- and two-particle states are generated by

$$\hat{s}(\vec{x})\Omega,\quad \hat{s}(\vec{x})\hat{s}(\vec{y})\Omega.$$

In the even case we can detect the presence of bound states by determining the decay rate of the CF associated with the vacuum subtracted states of the form

$$\hat{s}(\vec{x})\hat{s}(\vec{y})\Omega-(\Omega,\hat{s}(\vec{x})\hat{s}(\vec{x})\Omega)\Omega\equiv f(\vec{x},\vec{y})\Omega.$$

These states are orthogonal to the one-particle states. The associated partially truncated CF is, by the FK formula,

$$\begin{aligned}&\langle s(x_1)s(x_2)s(x_3)s(x_4)\rangle-\langle s(x_1)s(x_2)\rangle\langle s(x_3)s(x_4)\rangle\\&\quad=(f(\vec{x}_1\vec{x}_2)\Omega,e^{-H|\tau|}f(\vec{x}_3,\vec{x}_4)\Omega),\end{aligned}\tag{1.2}$$

where $x_{10}=x_{20}=t_1$, $x_{30}=x_{40}=t_2$, and $\tau=t_2-t_1$. By the spectral representation of H a decay rate in $|\tau|$ of less than $2m$, where m is the mass of the one-particle state, signals the presence of a bound state. If we try the same thing in the odd case, then one-particle intermediate states (which are not orthogonal to the two-particle states) on the right hand side of Eq. (1.2) gives us only a one-particle decay. The way around this is well known in Euclidean QFT, and is known as Euclidean subtraction (see Refs. [6,9,10]). In our context, instead of the left hand side of Eq. (1.2) we consider the CF

$$\begin{aligned}F(x_1x_2x_3x_4)=&\langle\overline{s}(x_1)\overline{s}(x_2)\overline{s}(x_3)\overline{s}(x_4)\rangle\\&-\langle\overline{s}(x_1)\overline{s}(x_2)\rangle\langle\overline{s}(x_3)\overline{s}(x_4)\rangle\\&+\sum_{y_1y_2\in Z^d}\langle\overline{s}(x_1)\overline{s}(x_2)\overline{s}(y_1)\rangle\Gamma(y_1,y_2)\\&\times\langle\overline{s}(y_2)\overline{s}(x_3)\overline{s}(x_4)\rangle,\end{aligned}\tag{1.3}$$

where $\overline{s}(x)=s(x)-\langle s(x)\rangle$ and $\Gamma(x,y)$ is minus the convolution inverse of the truncated two-point function $S(x,y)=\langle\overline{s}(x)\overline{s}(y)\rangle$, i.e.,

$$\sum_{z\in Z^d}\Gamma(x,z)S(z,y)=-\delta_{xy}.$$

We note that $\langle\overline{s}(x_1)\overline{s}(x_2)\overline{s}(x_3)\rangle=\langle s(x_1)s(x_2)s(x_3)\rangle^T$, so that the sum in Eq. (1.3) converges and for the four-point function

$$\begin{aligned}&\langle\overline{s}(x_1)\overline{s}(x_2)\overline{s}(x_3)\overline{s}(x_4)\rangle-\langle\overline{s}(x_1)\overline{s}(x_2)\rangle\langle\overline{s}(x_3)\overline{s}(x_4)\rangle\\&\quad-\langle\overline{s}(x_1)\overline{s}(x_3)\rangle\langle\overline{s}(x_2)\overline{s}(x_4)\rangle+\langle\overline{s}(x_1)\overline{s}(x_4)\rangle\\&\quad\times\langle\overline{s}(x_2)\overline{s}(x_3)\rangle=\langle s(x_1)s(x_2)s(x_3)s(x_4)\rangle^T.\end{aligned}$$

Indeed F will be seen to have a greater than one-particle decay, roughly a two-particle decay, but the price we pay is that we do not know of a spectral representation for F, i.e., we do not know the relation of the decay to the spectrum of H. What is done is to use another representation for Γ, which involves F, in conjunction with a Bethe-Salpeter equation for F. Before giving this representation we first point out that the truncated two-point function $S(x,y)$ admits a lattice version of the Kallen-Lehman representation (see Refs. [2,3]), and that singularities of $\tilde{S}(p)$, the Fourier transform of S, on the positive imaginary p_0 axis are points in the *e-m* spectrum. In terms of $\tilde{\Gamma}(p)=-\tilde{S}(p)^{-1}$ this mean that zeros of $\tilde{\Gamma}(p)$, with $p=(i\chi,\vec{p})$, and $\vec{p}\in T_{d-1}\equiv[-\pi,\pi]^{d-1}$, are in the *e-m* spectrum. Throughout this paper the Fourier transform is defined without factors of 2π. As $\tilde{\Gamma}(p)$ is an increasing function of χ, $\tilde{\Gamma}(p)$ is infinite for some point between consecutive discrete zeros.

The representation for Γ that we use is given in operator form by $\Gamma=M+LFL'$, or in terms of kernels by

$$\begin{aligned}\Gamma(x,y)=M(x,y)+\int L(x;x_1x_2)F(x_1x_2x_3x_4)L'(x_3x_4;y)\\\times dx_1dx_2dx_3dx_4,\end{aligned}\tag{1.4}$$

where, denoting $G(y_1y_2y_3)\equiv\langle s(y_1)s(y_2)s(y_3)\rangle^T$,

$$L(x;x_1x_2)=\int\Gamma(x,y')G(y'x_3'x_4')\Phi(x_3'x_4'x_1x_2)dy'dx_3'dx_4',$$

$$\begin{aligned}&L'(x_3x_4,y)\\&\quad=\int\Phi(x_3x_4x_1'x_2')G(x_1'x_2'u')\Gamma(u'y)dx_1'dx_2'du',\end{aligned}$$

where we use an integral notation for lattice sums. The various coordinates and sums are restricted by $x_{10}=x_{20}$, $x_{30}=x_{40}$, $x_{10}'=x_{20}'$, and $x_{30}'=x_{40}'$. $\Phi(x_1x_2x_3x_4)$ is minus the matrix inverse of F considering $F(x_1x_2x_3x_4)$ as a matrix operator in $s\ell_2(A)$, and a symmetric subspace of $\ell_2(A)$ where $A=\{(x_1,x_2)\in Z^{2d}/x_{10}=x_{20}\}$; $M=\Gamma-LFL'$. Equation (1.4) is the first in a hierarchy of equations to be used in an analysis of higher points in the spectrum, and versions of this representation were used in Refs. [6,9].

It turns out that F has roughly a two-particle decay; M, Φ, L, and L' have three-particle decay. This can be shown using decoupling of hyperplane methods (see Refs. [4,9,10]). We give the setup for the method and do some sample calculations in Appendix C. We give an intuitive picture for these decay rates in terms of cancellation of singularities in the Fourier transform. We denote by m the singularity of $\tilde{\Gamma}(i\chi,\vec{p}=\theta)$ for χ between m and the bound state mass m_b. For F, a singularity at m is introduced by the subtraction, but for the inverse $-\Phi$ there is no singularity and thus Φ has three-particle decay. For $L=\Gamma G\Phi G$ has a spectral representation, and the singularity of G at m is canceled by the zero of Γ at m. The singularity of Γ at m is canceled by the zero of Φ. Thus L has a three-particle decay, so does L'. Considering $M=\Gamma-LFL'$ the subtraction cancels the singularity of Γ at m, thus giving three-particle decay for M. The momentum space form of Eq. (1.4) in conjunction with a BS equation for F will be used to analyze the bound state problem. All the kernels appearing in Eq. (1.4) are translationally invariant. In Appendix A we introduce a newly devised system of lattice relative coordinates (see Refs. [1,3]), and show that the Fourier transform of Eq. (1.4), written in relative coordinates, is given by

$$\tilde{\Gamma}(k)=\tilde{M}(k)+(2\pi)^{-2(d-1)}\int \tilde{L}(\vec{p},k)\tilde{F}(\vec{p},\vec{q},k)\times\tilde{L}'(\vec{q},k)d\vec{p}\,d\vec{q}. \tag{1.5}$$

The BS equation is $F=D_0+D_0KF$ or, in terms of kernels, with $x_{10}=x_{20}$, $x_{30}=x_{40}$,

$$\begin{aligned}F(x_1x_2x_3x_4)=&D_0(x_1x_2x_3x_4)\\&+\int D_0(x_1x_2y_1y_2)K(y_1y_2y_3y_4)\\&\times F(y_3y_4x_3x_4)dy_1dy_2dy_3dy_4,\end{aligned} \tag{1.6}$$

where

$$D_0(x_1x_2x_3x_4)=\langle\overline{s}(x_1)\overline{s}(x_3)\rangle\langle\overline{s}(x_2)\overline{s}(x_4)\rangle+\langle\overline{s}(x_1)\overline{s}(x_4)\rangle\times\langle\overline{s}(x_2)\overline{s}(x_3)\rangle,$$

and the sums in Eq. (1.6) are restricted by $y_{10}=y_{20}$, $y_{30}=y_{40}$. In Appendix B we show that in momentum space, conjugate to the relative coordinates for Eq. (1.6), we can write

$$\begin{aligned}\tilde{F}(\vec{p},\vec{q},k)=&\tilde{D}_0(\overline{p},\overline{q},k)\\&+(2\pi)^{-(d-1)}\int \tilde{D}_0(\vec{p},\vec{p}\,',k)\tilde{K}(\vec{p}\,',\vec{q}\,',k)\\&\times\tilde{F}(\vec{q}\,',\vec{q},k)d\vec{p}\,'\,d\vec{q}\,'.\end{aligned} \tag{1.7}$$

In the ladder approximation K is local, and Eq. (1.7) can be solved for $\tilde{F}$. All this looks rather formidable, but it turns out that due to good spatial decay properties of the kernels in Eqs. (1.5) and (1.7) that only points in the space kernels that are separated by one unit or less play a dominant role and allow us to determine the zeros of $\tilde{\Gamma}(i\chi,\vec{p})=0$ for $(2-\varepsilon')m<\chi<2m$ and $\varepsilon'>0$, thus obtaining our result.

We now describe the organization of the paper. In Sec. II we determine the solution of the BS equation (1.7) in the ladder approximation. Our spectral results are obtained in Sec. III. Concluding remarks are made in Sec. IV. We have included three appendixes. In Appendix A we introduce newly devised lattice relative coordinates and derive the Fourier transform of the BS equation. In appendix B we derive the representation equation (1.5) for $\tilde{\Gamma}(k)$. We set up the decoupling of hyperplane scheme, and do sample calculations for decay of the CF in Appendix C.

II. BETHE-SALPETER EQUATION

Here we obtain the BS kernel in the ladder approximation, and a representation for the solution F. A fact that we use throughout is that at $\beta=0$ only coincident points of a truncated function contribute as follows from the exponential tree decay of a truncated CF. We will consider various matrix operators below and their inverses. For matrix operators $M(x_1x_2x_3x_4)$, $x_{10}=x_{20}$, and $x_{30}=x_{40}$, acting in $s\ell_2(A)$, we decompose into a diagonal part M_d and a nondiagonal part M_n, i.e., $M=M_d+M_n$ and the inverse is defined by $M^{-1}=(1+M_d^{-1}M_n)^{-1}M_d^{-1}$; the nondiagonal parts will be at least of order β.

Also, in what follows we use the moment identities

$$\langle\overline{s^3}\rangle=\langle s^3\rangle-3\langle s^2\rangle\langle s\rangle+2\langle s\rangle^3,$$

$$\langle\overline{s^4}\rangle=\langle s^4\rangle-4\langle s^3\rangle\langle s\rangle+6\langle s^2\rangle\langle s\rangle^2-3\langle s\rangle^4.$$

We can write

$$F=D+C$$

where

$$D(x_1x_2x_3x_4)=\langle\overline{s}(x_1)\overline{s}(x_2)\overline{s}(x_3)\overline{s}(x_4)\rangle-\langle\overline{s}(x_1)\overline{s}(x_2)\rangle\times\langle\overline{s}(x_3)\overline{s}(x_4)\rangle,$$

$$C(x_1x_2x_3x_4)=\sum_{y_1,y_2}\langle\overline{s}(x_1)\overline{s}(x_2)\overline{s}(y_1)\rangle\Gamma(y_1y_2)\times\langle\overline{s}(y_2)\overline{s}(x_3)\overline{s}(x_4)\rangle.$$

Note that F can also be written as

$$F=\langle s(x_1)s(x_2)s(x_3)s(x_4)\rangle^T+D_0+C$$

where

$$D(x_1x_2x_3x_4)=\langle\overline{s}(x_1)\overline{s}(x_3)\rangle\langle\overline{s}(x_2)\overline{s}(x_4)\rangle+\langle\overline{s}(x_1)\overline{s}(x_4)\rangle\times\langle\overline{s}(x_2)\overline{s}(x_3)\rangle.$$

Expanding in β we find

$$D_d(x_1x_2x_3x_4)=(\langle s^4\rangle-\langle s^2\rangle^2-4\langle s\rangle\langle s^3\rangle+8\langle s^2\rangle\langle s\rangle^2-4\langle s\rangle^4 \times\delta(x_3-x_1)\,\delta(x_4-x_2)\,\delta(x_1-x_2) +\langle\overline{s^2}\rangle\,\delta(x_3-x_1)\,\delta(x_4-x_2)(1-\delta(x_1-x_2)) +0(\beta)),$$

$$\Gamma(u,\nu)=-\langle\overline{s^2}\rangle^{-1}\delta(u-\nu),$$

$$\langle\overline{s(x_1)}\overline{s(x_2)}\overline{s(x_3)}\rangle=\langle s(x_1)s(x_2)s(x_3)\rangle^T =\langle\overline{s^3}\rangle\,\delta(x_3-x_1)\,\delta(x_1-x_2)+0(\beta),$$

$$C_d(x_1x_2x_3x_4)=\langle-\overline{s^2}\rangle^{-1}\langle\overline{s^3}\rangle^2\,\delta(x_3-x_1)\,\delta(x_4-x_2) \times\delta(x_1-x_2)+0(\beta),$$

$$D_{od}(x_1x_2x_3x_1)=2\langle\overline{s^2}\rangle^2\,\delta(x_1-x_3)\,\delta(x_2-x_4)\,\delta(x_1-x_2) +\langle\overline{s^2}\rangle^2\,\delta(x_1-x_3)\,\delta(x_2-x_4) \times(1-\delta(x_1-x_2))+O(\beta).$$

Thus

$$K=D_0^{-1}-F^{-1}=D_{0d}^{-1}-F_d^{-1}+0(\beta)=D_{0d}^{-1}-(D_d+C_d)^{-1} +O(\beta)$$
$$=\rho\,\delta(x_1-x_2)\,\delta(x_1-x_3)\,\delta(x_2-x_4)+0(\beta)$$
$$\equiv K_L+O(\beta),$$

$$\rho=(2\langle\overline{s^2}\rangle^2)^{-1}(\langle\overline{s^4}\rangle-3\langle\overline{s^2}\rangle^2-\langle\overline{s^3}\rangle^2/\langle\overline{s^2}\rangle)(\langle\overline{s^4}\rangle-\langle\overline{s^2}\rangle^2 -\langle\overline{s^3}\rangle^2/\langle\overline{s^2}\rangle)^{-1}=(2\langle\overline{s^2}\rangle^2)^{-1}\alpha,$$

where we have defined K_L, the ladder approximation, which is local, and $\tilde{K}_L(\vec{p},\vec{q},k)=\rho$.

We now obtain an approximation for

$$(g,\tilde{F}f)=(g,\overline{D_0}(1-(2\pi)^{-2(d-1)}\tilde{K}\tilde{D}_0)^{-1}f)$$

for g and f even functions of $\overline{p}$, i.e., $f(\vec{p})=f(-\vec{p})$ and $k=(k_0,\vec{k}=\vec{\sigma})$. For $\tilde{D}_0 f$ we have

$$\tilde{D}_0 f(\overline{p})=2(2\pi)^{d-2}\int\tilde{S}(k_0-p_0,\vec{p})\tilde{S}(p_0,\vec{p})dp_0 f(\vec{p}) \equiv G_0(\vec{p},k)f(\vec{p}). \tag{2.1}$$

Replacing $\tilde{K}$ by $\tilde{K}_L$, expanding $(1-\tilde{K}_L\tilde{D}_0)^{-1}$, evaluating and resuming we find

$$(g,\tilde{F}f)=\int\overline{g}(\vec{p})G_0(\vec{p},k)f(\vec{p})d\vec{p} +\frac{\rho'}{1-\rho' I}\left(\int\overline{g}(\vec{p})G_0(\vec{p},k)d\vec{p}\right) \times\left(\int G_0(\vec{q},k)f(\vec{q})d\overline{q}\right), \tag{2.2}$$

where $\rho'=(2\pi)^{-2(d-1)}\rho$ and $I=\int G_0(\vec{p},k)d\vec{p}$.

Setting $k_0=(i\chi,\vec{k}=\vec{0})$ we now determine the behavior of I as a function of the spectral parameter χ or ε where $\varepsilon=2m-\chi$. These properties play an important role in the analysis of the spectrum in Sec. III.

The general representation for $\tilde{S}(p)$, the Fourier transform of S, can be obtained by adapting the work of Refs. [2–4] to show that

$$\tilde{S}(p_0,\vec{p})=\frac{Z(\vec{p})}{\cosh\omega(\vec{p})-\cos p_0}+\int_{\cosh\overline{m}}^{\infty}\frac{d\eta(a,\vec{p})}{a-\cos p_0}, \tag{2.3}$$

where $\overline{m}=-(2-\varepsilon')\ln\beta$, $d\eta$ is a positive measure, and $Z(\vec{p}),\omega(\vec{p})$ are real analytic in $\overline{p};\overline{m}$ is lower bound for the spectrum above one particle. $\omega(\vec{p})$ is the one-particle dispersion function $\omega(\vec{p})\geqslant\omega(\vec{0})=m$. Explicitly,

$$\omega(\vec{p})=-\ln\beta-\ln\langle\overline{s^2}\rangle-\beta 2d\langle s\rangle\langle\overline{s^2}\rangle^{-1}\langle\overline{s^3}\rangle-2\beta(d-1)\langle\overline{s^2}\rangle +\beta\langle\overline{s^2}\rangle 2\sum_{i=1}^{d-1}(1-\cos p_i)+0(\beta^2),$$

$$Z(\vec{p})\beta=(2\beta)^{-1}+0(1).$$

Using this representation we can explicitly perform the p_0 integral in $G_0(\vec{p},k)$. Only the product of the one-particle terms in Eq. (2.3) can give rise to a singularity in χ as the other terms are analytic in χ up to at least $-(3-\varepsilon')\ln\beta$. Keeping only these terms and denoting the result by I_1, we obtain

$$I_1=16(2\pi)^{d-2}\pi e^{-\chi} \times\int\frac{Z(\vec{p})^2}{\tanh w(\overline{p})}\,\frac{1}{(e^{-\chi}-e^{-2w(\vec{p})})(e^{2w(\vec{p})}-e^{-\chi})}d\vec{p}>0, \tag{2.4}$$

showing the analyticity in χ for $0<\mathrm{Re}\,\chi<2m$. Furthermore I_1, as well as I, is a strictly increasing function of χ for $0<\chi<2m$. To see the behavior of I_1 as $\chi\uparrow 2m$, we write $\chi=2m-\varepsilon$, and using $w(\vec{p})=m+\beta<\overline{s^2}>\overline{p}^2+0(\beta^2)$ we have, for small β,

$$I_1=2(2\pi)^{2_d-2}(1-e^{-\varepsilon})^{-1}\langle s^2\rangle^2. \tag{2.5}$$

III. SPECTRAL RESULTS

Recall that we are looking for zeros of $\tilde{\Gamma}(k)$ with $k=(k_0=i\chi,\vec{k}=\vec{0})$, $(2-\varepsilon')m<\chi<2m$, where $m\sim-\ln\beta$ and it is convenient to write $\chi=2m-\varepsilon$, $\varepsilon>0$. We consider the dominant contributions to the terms in Eq. (1.5) for small β. In what follows we will state spatial decay properties of M, L, and L'. These properties can be obtained by the decoupling of hyperplane method (see Refs. [4,9,10] and Appendix C). For $\tilde{M}(k)$, k is the momentum conjugate to the relative coordinate $\tau=y-x$. We denote the partial Fourier transform in the space variables by ^. We find

$$\hat{M}(\tau_0)=(M(0,0)=\Gamma(0,0)-L(000)F(0000)L'(000)) +0(\beta),\quad\tau_0=0;$$

$$\hat{M}(\tau_0)=M(0,e_0)\beta+0(\beta^2),\quad M(0,e_0)=1,\quad |\tau_0|=1;$$

$$|\hat{M}(\tau_0)|\leqslant c\beta^{3|\tau_0|},\quad |\tau_0|\geqslant 2.$$

Thus, in the sum $\Sigma_{\tau_0}e^{-ik_0\tau_0}\hat{M}(\tau_0)$, the dominant term comes from $|\tau_0|=1$ and is $\beta\cosh\chi\sim e^{-\varepsilon}/\beta$. For $L(xx_1x_2)$, where $x_{10}=x_{20}$, we let $d=|x_{10}-x_{20}|$ and find $|L(xx_1x_2)|\leqslant c\beta^{3d}$, $d\geqslant 1$ and $L(xx_1x_2)=L(xxx)+0(\beta)$, $d=0$; similar results held for $L'(x_3x_4,y)$, where $x_{30}=x_{40}$. We explain the $d=0$ property of $L(xx_1x_2)$, referring to the definition in Eq. (1.4). For $\beta=0$, Γ and G are nonzero only for coincident points. To see that only coincident points enter into Φ, we consider F. For $\beta=0$, F only has a diagonal part (which is not local). However, if the first two (or last two) points coincide then only for all coinciding points will F have a nonzero value. This property carries over to $\Phi=-F^{-1}$. By a subscript zero we denote the values of kernels at coincident points. We denote $M(0,0)\equiv M_0$, $\Gamma(00)\equiv\Gamma_0$, $G(000)=G_0$, $F(0000)\equiv F_0$, $\Phi(0000)=\Phi_0$, $L(000)=L_0$, and $L'(000)\equiv L'_0$. Their values will be needed below. We find

$$F_0\langle\overline{s}^4\rangle-\langle\overline{s}^2\rangle^2-\langle\overline{s}^3\rangle^2\langle\overline{s}^2\rangle^{-1}=-\Phi_0^{-1},$$

$$M_0=-\langle\overline{s}^2\rangle^{-1}\langle\overline{s}^2\rangle^{-}(\langle\overline{s}^3\rangle^2(\langle\overline{s}^4\rangle-\langle\overline{s}^2\rangle^2\langle\overline{s}^3\rangle^2\langle\overline{s}^2\rangle^{-1}),$$

$$L_0=\langle\overline{s}^2\rangle^{-1}\langle\overline{s}^3\rangle(\langle\overline{s}^4\rangle-\langle\overline{s}^2\rangle^2-\langle\overline{s}^3\rangle^2\langle\overline{s}^2\rangle^{-1})^{-1}=L'_0.$$

In what follows recall that we are assuming $L_0\neq 0$. Keeping the dominant terms for the equation we can write $\tilde{\Gamma}(k_0=i\chi,\tilde{k}=0)\equiv\tilde{\Gamma}(\chi)=0$, letting $\tilde{F}(\vec{p},\vec{q},k=(i\chi,\vec{0}))\equiv\tilde{F}(\vec{p},\vec{q},\chi)$ by abuse of notation, the approximate condition

$$\tilde{\Gamma}(\chi)=0=M_0+\frac{e^{-\varepsilon}}{\beta}+(2\pi)^{-2(d-1)}L_0^2\int\tilde{F}(\vec{p},\vec{q},\chi)d\vec{p}\,d\vec{q}.$$

Using Eq. (2.2) with $g=f=1$ for the last integral we have the approximate condition

$$\tilde{\Gamma}(\chi)=0=M_0+\frac{e^{-\varepsilon}}{\beta}+L_0^2\left(\frac{1}{1-\rho' I}\right),\quad \chi=2m-\varepsilon. \tag{3.1}$$

In Eq. (3.1) M_0 may be negative, for example, for a *ssd* with a small odd part. However, for small β the positive second term will dominate M_0. Recalling the properties of I, i.e., positive and strictly increasing in χ for $0<\chi<2m$, in Sec. II we see that if $\alpha<0$ then $\rho' I<0$ and there will be no solution. For $\alpha>0$, and β sufficiently small, $1-\rho' I=0$ is approximately $1-\rho' I_1=0$ which by Eq. (2.5) has the approximate solution $\varepsilon_0=-\ln(1-\alpha)$. We denote the solution of $1-\rho' I=0$ by ε, and it is given by $\varepsilon_1=\varepsilon_0+0(\beta)$. Thus for $\varepsilon<\varepsilon_1$, $1-\rho' I$ is negative, and for ε near ε_1 we have a solution ε_b of Eq. (3.1). Thus we have a bound state with mass m_b, where

$$2m-m_b=\varepsilon_b=-\ln(1-\alpha)+O(\beta).$$

We note that $\tilde{\Gamma}$ goes to infinity as $\varepsilon\to\varepsilon_1$, which is a remnant (in this approximation for $\tilde{\Gamma}$) of the infinity that $\tilde{\Gamma}$ must assume between consecutive zeros, of mass m, and of the bound state mass [11].

IV. CONCLUDING REMARKS

We have found a simple criterion for the existence of a bound state for small β. The question arises as to the location and number of bound states when β is not small. Also there is the analytic problem of whether or not $2m-m_b$ admits an analytic extension in β as is the case for $m+\ln\beta$. The existence of weakly bound states in lattice gauge and gauge-matter models (strongly bound states are present) is also an open question, and the methods developed here open a way to treat these problems.

APPENDIX A: REPRESENTATION OF Γ

We develop a Fourier transform representation for Γ. For $x,y\in Z^d$, we write

$$\Gamma(x,y)=M(x,y)+\int L(xy_1y_2)\,\delta(y_2^0-y_1^0)F(y_1y_2y_3y_4)\times\delta(y_4^0-y_3^0)L'(y_3y_4y)dy_1dy_2dy_3dy_4, \tag{A1}$$

where the kernels are translationally invariant. Introduce the variable $\tau=y-x$ and the conjugate variable k and a bar notation for functions of the relative coordinates.
We let

$$\alpha=u_2-u_1,\quad \beta=u_1-u_3,\quad \gamma=u_3-u_2,$$

and define $\overline{F}$, $\overline{L}$, and $\overline{L'}$ by

$$\overline{F}(\alpha,\beta,\gamma)=F(0,u_2-u_1=\alpha,u_3-u_1=\alpha+\gamma,u_4-u_1=\alpha+\beta+\gamma),$$

$$\overline{L}(\beta,\gamma)=L(0,u_3-u_2=\gamma,u_4-u_2=\beta+\gamma), \tag{A2}$$

$$\overline{L}(\alpha,\gamma)=L'(0,u_2-u_1=\alpha,u_3-u_2=\gamma),$$

which are associated with $F(u_1u_2u_3u_4)$, $L(u_2u_3u_4)$, and $L'(u_1u_2u_3)$, respectively.

Introduce the variables (see [11])

$$\xi'=y_2-y_1,\quad \eta'=y_4-y_3,\quad \tau'=y_1-x,\quad \tau''=y-y_4$$

which imply

$$y_1=\tau'+x_1,\quad y_2=\xi'+y_1=\xi'+\tau'+x,$$

$$y_3=y_4-\eta'=y-\tau'-\eta',\quad y_4=y-\tau''.$$

We have the following symmetry properties: $L(u_2u_3u_4)$ is invariant under $u_3\leftrightarrow u_4$, $L'(u_1u_2u_3)$ is invariant under $u_1\leftrightarrow u_2$, and $F(u_1u_2u_3u_4)$ is invariant under $u_1\leftrightarrow u_2$ and $u_3\leftrightarrow u_4$. Thus for $\overline{F}$ we have

$$\overline{F}(\alpha,\beta,\gamma)=\overline{F}(-\alpha,\beta,\gamma+\beta)=\overline{F}(\alpha,-\beta,\gamma+\beta). \tag{A3}$$

Now, since $y_3-y_2=(y-\tau''-\eta')-(\xi'+\tau'+x)=y-x-\tau'-\tau''\xi'-\eta'$, and using the symmetry property [Eq. (A3)], we have

$$F(y_1y_2y_3y_4)=\overline{F}(\xi',\eta',\tau-\tau'-\tau''-\varepsilon'-\eta')$$
$$=\overline{F}(-\xi',\eta',\tau-\tau'-\tau''),$$

$$L(xy_1y_2)=\overline{L}(\xi',\tau'),\quad L'(y_3y_4y)=\overline{L'}(\eta',\tau'').$$

Thus for Eq. (A1) we have

$$\overline{\Gamma}(\tau)=\overline{M}(\tau)-\int d\overline{\xi}\,d\overline{\eta'}\,d\tau'\,d\tau''\overline{L}(\vec{\xi}',\tau')\times\overline{F}(-\vec{\xi}',-\vec{\eta}',\tau-\tau'-\tau'')\overline{L'}(\vec{\eta}',\tau''). \tag{A4}$$

We define the Fourier transform of a function $f(x)$ with a conjugate variable p with a factor e^{-ipx} and no factor of 2π. Taking the Fourier transform of Eq. (A4) and dropping the bars gives

$$\tilde{\Gamma}(k)=\tilde{M}(k)+(2\pi)^{-2(d-1)}\times\int d\vec{p}\,d\vec{q}\tilde{L}(\vec{p},k)\tilde{F}(\vec{p},\vec{q},k)\tilde{L}'(\vec{q},k)$$

which is our desired result [Eq. (1.5)].

APPENDIX B: BS EQUATION

Here we develop a BS equation for F. For $F(x_1x_2x_3x_4)$, $x_{10}=x_{20}$ and $x_{30}=x_{40}$. We have

$$F(x_1x_2x_3x_4)=D_0(x_1x_2x_3x_4)+\int D_0(x_1x_2y_1y_2)\delta(y_{10}-y_{20})K(y_1y_2y_3y_4)\times F(y_3y_4x_3x_4)\delta(y_{30}-y_{40})dy_1dy_2dy_3dy_4. \tag{B1}$$

We emphasize that the kernels are only defined for equal times in the first two and last two points. We introduce variables as in the even case (see [11]) by

$$\xi=x_2-x_1,\quad \eta=x_4-x_3,\quad \tau=x_3-x_2,\quad \xi_0=\eta_0=0.$$

Then using a bar for functions of the relative variables, we have

$$\overline{F}(\vec{\xi},\vec{\eta},\tau)=\overline{D}_0(\vec{\xi},\vec{\eta},\tau)-\int d\vec{\xi}'d\vec{\eta}'d\tau'd\tau''\overline{F}(\vec{\xi},\vec{\xi}',\tau')\times\overline{K}(-\vec{\xi}',-\vec{\eta}',\tau-\tau'-\tau'')\overline{D}_0(\vec{\eta}',\vec{\eta},\tau''). \tag{B2}$$

where we have used the $x_1\leftrightarrow x_2$ and $x_3\leftrightarrow x_4$ symmetry properties of F, D_0, and K. Denoting the conjugate variables to $\vec{\xi}$, $\vec{\eta}$, τ by $\vec{p}$, $\vec{q}$, and k, respectively, and taking the Fourier transform of Eq. (B2), gives (dropping the bar)

$$\tilde{F}(\vec{p},\vec{q},k)=\tilde{D}_0(\vec{p},\vec{q},k)+(2\pi)^{-2(d-1)}\int d\vec{p}'d\vec{q}'\tilde{F}(\vec{p},\vec{p}',k)\times\tilde{K}(\vec{p}',\vec{q}',k)\tilde{D}_0(\vec{q}',\vec{p},k), \tag{B3}$$

which is our desired result [Eq. (1.7)].

APPENDIX C: DECAY OF CORRELATIONS. DECOUPLING OF THE HYPERPLANE METHOD

Here, by a sample calculation, we show how the decoupling of the hyperplane method is used to obtain a decay of the CF introduced in Sec. I (see also Refs. [4,9]). In particular we obtain the temporal falloff of F and Φ. The same method applies for spatial falloff and for L, L', and M.

For the truncated function

$$G_{\theta,\eta}(x,y)=\langle\theta(x)\eta(y)\rangle-\langle\theta(x)\rangle\langle\eta(y)\rangle, \tag{C1}$$

we introduce a duplicate variable representation depending on the hyperplane decoupling complex parameters $\{w_q\}$ where w_q replaces β for the bonds connecting the $x_0=q$ and $x_0=q+1$ hyperplanes. The representation is, with $x_0\leq q<y_0$,

$$G_{\theta,\eta}(x,y)\{w_q\})=\int[\theta(x)-\theta'(x)][\eta(y)-\eta'(y)]\times\exp\left[\sum_{q,z_0=q}w_q(s(z)s(z+e_0)+s'(z+e_0))\right]\times\exp[S(s)+S(s')]d\mu(s')]d\mu(s')/2D, \tag{C2}$$

where D is the normalization factor. The primes in θ' and η' mean functions of the duplicate variables s'. $S(s)$ is the action for the remaining bonds. By the polymer expansion (see Ref. [4]) $G_{\theta,\eta}$ is jointly analytic in $\{w_q\}$ for small $|w_q|$, and equal to Eq. (C1) for all $w_q=\beta$. From now on we suppress the $\{w_q\}$ from the notation of $G_{\theta,\eta}$ in Eq. (C2). The way in which decay is obtained is to show the vanishing of $G_{\theta,\eta}$ or of w_q derivatives at $w_q=0$. From the joint analyticity in $\{w_q\}$ and Cauchy bounds for derivatives, the decay follows.

As we will be calculating w_q derivatives of F and Φ, it is convenient to have a general formula. Expanding the numerator and denomiator of Eq. (C2) in powers of w_q, we find

$$\begin{aligned}2G_{\phi,\psi}(x,y)&=2\sum_{z_0=q}\{G_{\phi,s}(x,z)G_{s,\psi}(z+e_0,y)\}_{w_q=0}w_q\\&+\sum_{z_0=q,}\{G_{\phi,ss(\vec{u})}(x,y)G_{ss(\vec{u}),\psi}(z+e_0,y)\\&+G_{\phi,s\langle s(\overline{u})-s(\overline{u})\langle s\rangle}(x,z)G_{s\langle s(\vec{u})\rangle-s(\overline{u})\langle s\rangle,\psi}(z+e_0,y)\\&-4G_{\phi,s\langle s(\vec{u})\rangle}G_{s\langle s(\vec{u})\rangle,\psi}(z+e_0,y)\}_{w_q=0}w_q^2+0(w_q^3),\end{aligned} \tag{C3}$$

where $G_{\phi,ss(\vec{u})}(x,z)=\langle\phi(x)s(z)s(z+\vec{u})\rangle-\langle\phi(x)\rangle\langle s(z)s(z+\vec{u})\rangle$, etc. For notational simplicity we write the w_q^2 coefficient as, with $k=1$, 2, and 3,

$$\sum_{z_0=q,k} G_{\phi,\phi_k}(x,z)G_{\phi_k,\psi}(z+e_0,y). \tag{C4}$$

Specializing to $\phi=\psi=s$, denoting $\partial^r/\partial w_q^r$ by ∂^r, $\partial^r/\partial w_q^r|w_{q=0}$ by ∂_0^r and $G_{s,s}$ by G, we have

$$\partial_0^0 G(x,y)=0, \tag{C5a}$$

$$\partial_0^1 G(x,y)=\sum_{z_0=q} G(x,z)G(z+e_0,y)|w_{q=0}, \tag{C5b}$$

$$\partial_0^0 \Gamma(x,y)=0, \tag{C5c}$$

$$\partial_0^1 \Gamma(x,y)=\delta_{x+e_0,y} \tag{C5d}$$

where we use $\partial\Gamma=\Gamma\partial G\Gamma$, which follows from $\Gamma G=-1$. As the arguments used in obtaining $\partial\Gamma$ will occur repeatedly in the sequel, we give more detail. We have

$$\partial\Gamma(x,y)=\sum_{u,v}\Gamma(x,u)\partial G(u,v)\Gamma(u,y);$$

by the properties of Γ, the u and v sums can be restricted by $u_0\le p$ and $v_0>p$, which allows us to use Eq. (C5b). Lifting the restrictions and using $\Gamma G=-1$ we obtain Eq. (C5d). Equations (C5a), (C5c), and (C5d) imply the decay

$$|G(x,y)|\le c\left|\frac{\beta}{c_1}\right|^{|y_0-x_0|}, \quad |\Gamma(x,y)|\le c\beta\left|\frac{\beta}{c_1}\right|^{2|y_0-x_0-1|}.$$

We now consider

$$F_{\phi,\psi}(x,y)=G_{\phi,\psi}(x,y)+\sum_{u,v} G_{\phi,s}(x,y)\Gamma(u,v)G_{s,\psi}(v,y), \tag{C6}$$

and note that, if ϕ(or ψ) is a constant or proportional to s, then $F_{\phi,\psi}=0$. Using Eqs. (C3) and (C5c), we see that $F_{\phi,\psi}(x,y)|_{w_{q=0}}$, and for the first derivative we have

$$\partial F_{\phi,\psi}=\partial G_{\phi,\psi}+\partial G_{\phi,s}\Gamma G_{\phi,s}\partial\Gamma G_{s,\psi}+G_{\phi,s}\Gamma\partial G_{s,\psi}. \tag{C7}$$

Restricting sums and using Eq. (C3) gives

$$\begin{aligned}
&\partial_0 F_{\phi,\psi}(x,y)\\
&\quad=\sum_{z_0=q}\Big[G_{\phi,s}(x,z)G_{s,\psi}(z+e_0,y)\\
&\qquad+\sum_{u_0,v_0>q} G_{\phi,s}(x,z)G(z+e_0,u)\Gamma(u,v)G_{s,\psi}(v,y)\\
&\qquad+\sum_{u\le q,v_0>q} G_{\phi,s}(x,v)\partial\Gamma(u,v)G_{s,\psi}(v,y)\\
&\qquad+\sum_{u_0,v_0\le q} G_{\phi,s}(x,u)\Gamma(x,v)G(v,z)G_{s,\psi(z+e_0,y)}|_{w_q=0}.
\end{aligned} \tag{C8}$$

Thus we have the decay

$$|F_{\phi,\psi}(x,y)|\le c\left|\frac{\beta}{c'}\right|^{2|y_0-x_0|}.$$

For the second derivative, writing $\partial^2\Gamma=\Gamma\partial^2 G\Gamma+2\Gamma\partial G\Gamma$, we have

$$\begin{aligned}
\partial^2 F_{\phi,\psi}=&\partial^2 G_{\phi,\psi}+\partial^2 G_{\phi,s}G_{s,\psi}+G_{\phi,s}\Gamma\partial^2 G\Gamma G_{s,\psi}\\
&+G_{\phi,s}\Gamma\partial^2 G_{s,\psi}+2G_{\phi,s}\Gamma\partial G\partial\Gamma G_{s,\psi}\\
&+2\partial G_{\phi,s}\partial\Gamma G_{s,\psi}+2G_{\phi,s}\partial\Gamma\partial G_{s,\psi}\\
&+2G_{\phi,s}\Gamma\partial G_{s,\psi}.
\end{aligned} \tag{C9}$$

The reason for the splitting of $\partial^2\Gamma$ is that the first line of Eq. (C9), denoted by R, has a product structure of F's, and the remaining terms taken together vanish. In more detail, using Eq. (C3),

$$\begin{aligned}
R(x,y)|_{w_{q=0}}=&\sum_{k,z_0=q} G_{\phi,\phi_k(x,z)G_{\phi_{k,\psi}}(z+e_0,y)}\\
&+\sum_{u_0>q} G_{\phi,\phi_k}(x,z)G_{\phi_{k,s}}(z+e_0,u)\Gamma(u,v)\\
&\times G_{s,\psi}(v,y)+\sum_{u,u',v,v'} G_{\phi,s}(x,v)\Gamma(u,u')\\
&\times G_{s,\phi_k}(u',z)G_{\phi_{k,s}}(z+e_0,v)\Gamma(v,v')G_{s,\psi}(v',y)\\
&+\sum_{u,v} G_{\phi,s}(x,u)\Gamma(u,v)G_{s,\phi_k}(v,z)G_{\phi_{k,\psi}}\\
&\times(z+e_0,y)|_{w_{q=0}}\\
=&\sum_{k,z_0=q} F_{\phi,\phi_k}(x,z)F_{\phi_k,\psi}(z+e_0,y)|_{w_{q=0}}.
\end{aligned} \tag{C10}$$

Only the $k=1$ term contributes to Eq. (C10) and taking $\phi=ss(\vec{r})$ and $\psi=ss(\vec{r}\,')$ we have

$$\begin{aligned}
\partial_0^2 F_{ss(\vec{r}),ss(\vec{r}\,')}(x,y)=&\sum_{z_0=q,\vec{r}\,''} F_{ss(\vec{r}),ss(\vec{r}\,'')}(x,z)F_{ss(\vec{r}\,''),ss(\vec{r}\,')}\\
&\times(z+e_0,y)|_{w_{q=0}},
\end{aligned} \tag{C11}$$

and recalling that F with s's is the same as F with $\overline{s}$'s gives

$$\begin{aligned}
\partial_0^2 F_{\overline{s\,s(\vec{r})},\overline{s\,s(\vec{r}\,')}}(x,y)=&\sum_{z_0=q,\vec{r}\,''} F_{\overline{s\,s(\vec{r})},\overline{s\,s(\vec{r}\,'')}(x,z)}\\
&\times F_{\overline{s\,s(\vec{r}\,'')}\overline{s\,s(\vec{r}\,')}}(z+e_0,y)_{w_q=0}.
\end{aligned} \tag{C12}$$

Letting $\Phi_{\overline{s\,s(\vec{r})},\overline{s\,s(\vec{r}\,')}}(x,y)$ denote the negative inverse of $F_{\overline{s\,s(\vec{r})},\overline{s\,s(\vec{r}\,')}}(x,y)$ and using the properties of $F_{\phi,\psi}(x,y)$, namely, $F_{\phi,\psi}(x,y)|_{w_{q=0}}$, and $\partial F_{\phi,\psi}(x,y)=0$ and Eq. (C12) we obtain

$$\partial_0^0\Phi_{\overline{s\,s(\vec{r})},\overline{s\,s(\vec{r}\,')}}(x,y)=0, \tag{C13a}$$

$$\partial_0\Phi_{\overline{s\,s(\vec r)},\overline{s\,s(\vec r\,')}}(x,y)=0, \tag{C13b}$$

$$\partial_0^2\Phi_{\overline{s\,s(\vec r)},\overline{s\,s(r\vec r\,')}}(x,y)=\delta_{\vec r,\vec r\,'}\delta_{x+e_0,y}, \tag{C13c}$$

which implies the decay

$$|\Phi_{\overline{s\,s(\vec r)},\overline{s\,s(\vec r\,')}}(x,y)|\le c\beta^2\left|\frac{\beta}{c_1}\right|^{3|y_0-x_0-1|}.$$

Spatial decay is obtained by using complex coupling parameters associated with separating hyperplanes perpendicular to each of the component directions of the spatial coordinates.

Now we consider L, which we write as

$$L_{s,ss(\vec r)}(x,y)=\sum_{u,\nu,\vec r\,'}\Gamma(x,\nu)G_{\overline{s,s\,s(\vec r\,')}}(u,\nu)\Phi\overline{s\,s(\vec r)},\overline{s\,s(\vec r\,')}\times(\nu,y), \tag{C14}$$

and note that in Φ we can replace $\overline{s}$ by s and the same for the first $\overline{s}$ in $G_{\overline{s,s\,s(\vec r\,')}}$, which for simplicity we denote by G_3. For $x_0\le q\le y_0$ and at $w_q=0$, $L(x,y)=0$, using Eqs. (C4), (C13a), and (C3) for G_3. However, to calculate derivatives and use the derivative formula [Eq. (C3)] $\langle\overline{s}\rangle$ and hence $\overline{s}$ cannot appear in ϕ or ψ. Thus we replace the $\overline{s}$ by s in $\Phi_{\overline{s\,s},\overline{s\,s}}$, and use the decomposition

$$G_{s,\overline{s\,s(\vec r\,')}}(u,\nu)=[G_{s,ss(\vec r\,')}-G_{s,s\langle s(\vec r\,')\rangle}-G_{s,\langle s\rangle s(\vec r\,')}+G_{s,\langle s\rangle\langle s(\vec r\,')\rangle}](u,\nu),$$

where the last term is zero. We write $L=L+R$, where

$$L_{\vec r}(x,y)=\sum_{u,\nu,\vec r\,'}\Gamma(x,\nu)G_{s,ss(\vec r\,')}(u,\nu)\Phi_{ss(\vec r\,'),ss(\vec r)}(\nu,y), \tag{C15}$$

$$R_{\vec r}(x,y)=\sum_{\vec r\,'}[\langle s(x+\vec r\,')\rangle\Phi_{ss(\overline{\vec r\,'}),ss(\overline{\vec r})}(x,y)+\langle s(x-\vec r\,')\rangle\Phi_{ss(\vec r\,'),ss(\vec r)}(x-\vec r\,',y)] \tag{C16}$$

For $x_0\le q<y_0$ and $\partial_0^0L(x,y)=0$,

$$\partial_0L(x,y)=\left\{\sum_{\substack{u,\nu\\ z_0=0,\vec r\,'}}[\delta_{xz}\delta_{z+e,u}Gs,ss(\vec r\,')(u,\nu)\times\Phi_{ss(\vec r\,'),ss(\overline{r})}(\nu,y)+\Gamma(x,u)G_{s,s}(u,z)\times G_{s,ss(\vec r\,')}(z+e_0,\nu)\Phi_{ss(\vec r\,'),ss(\vec r)}(\nu,y)]+\sum_{u,\nu,\vec r\,'}\Gamma(x,u)G_{s,ss(\vec r\,')}(u,\nu)\partial\Phi_{ss(\vec r\,'),ss(\overline{r})}\times(\nu,y)\right\}\Bigg|_{w_q}=0. \tag{C17}$$

The first two terms in ∂_0L sum to zero, and as $\partial_0\Phi(\nu,y)=0$ for $\nu_0\le q$ we have $\partial_0L(x,y)=0$.

For the second derivative, suppressing indices, denoting $G_{s,ss(\overline{u})}$, etc. by G_3, and recalling that $\partial^2\Gamma=\Gamma\partial^2G\Gamma+2\partial\Gamma\partial G\partial\Gamma$, we have

$$\partial^2L=\Gamma\partial^2G\Gamma G_3\Phi+\Gamma\partial^2G_3\partial\Phi+\Gamma G_3\partial^2\Phi+2\Gamma\partial G\partial\Gamma G_3\Phi+2\Gamma\partial G_3\partial\Phi+2\partial\Gamma G_3\partial\Phi+2\partial\Gamma\partial G_3\Phi. \tag{C18}$$

At $w_q=0$ only the first three terms can give a nonzero contribution for $x_0<q<y_0$. Using Eq. (C3), the first two terms of Eq. (C18) are, for $x_0\le q<y_0$,

$$\sum_{\substack{u_0\le q,i,z_0=q\\ \nu_0>q,\vec r\,'}}\Gamma(x,u)G_{s,\phi i}(u,z)F_{\phi i,ss(\vec r\,')}(z+e_\varepsilon,\nu)\Phi_{ss(\vec r\,'),ss(\overline{r})}\times(\nu,y)|w_q=0$$

and using the properties of $F_{\phi,\psi}$ we can replace ϕ_i by $\phi_1=ss(\vec r\,'')$ so that the first three terms of Eq. (C18) sum to zero. Here we have used Eq. (C13c) for $\partial_0^2\Phi$.

Turning to R of Eq. (C16), the two terms are equal, as seen by using the symmetry property $F_{ss(\vec r\,'),ss(\overline{r})}(x,y)=F_{ss(\vec r\,'),s(\overline{r})s}(x,y)$. As $\Phi(x,y)=0$ and $\partial\Phi(x,y)=0$ for $x_0\le q<y_0$ and $w_q=0$, we have $\partial_0^0R\vec r(x,y)=0$, $\partial_0R\vec r(x,y)=0$, and

$$\partial_0^2R\vec r(x,y)=\sum_{\vec r\,'}[\langle s(x+\vec r\,')\rangle\partial^2\Phi_{ss(\vec r\,'),ss(\vec r)}(x,y)+\langle s(x-\vec r\,')\rangle\partial^2\Phi_{ss(\vec r\,'),ss(\vec r)}(x-\vec r\,',y)]|w_q=0,$$

and using Eq. (C13c) gives $\partial_0^2R\vec r(x,y)=2\langle s(x+\vec r)\rangle\delta_{x+e_0,y}$. Altogether we have, for $x_0\le q<y_0$,

$$\partial_0^2L_{s,ss(\vec r)}(x,y)=\left\{2\langle s(x+\vec r)\rangle\delta_{x+e_0,y}+2\sum_{z_0=q,\nu\overline{r}'}\delta_{x,z}\partial G_{s,ss(\vec r\,')}\times(z\quad+e_0,\nu)\Phi_{ss(\vec r\,'),ss(\vec r)}(\nu,y)+2\sum_{z_0=q,\nu',\vec r\,'}\delta_{x,z}\partial G(z+e_0,\nu')\Gamma(\nu',u)\times G_{s,ss(\vec r\,')}(u,\nu)\Phi_{ss(\vec r\,'),ss(\overline{r})}(\nu,y)\right\}\Bigg|_{w_q}=0,$$

so that $\partial_0^2L_{s,ss(\vec r)}(x,y)=0$ for $x_0<q<y_0$. Thus we have

$$|L_{s,ss(\vec r)}(x,y)|\le\begin{cases}c|\beta|c_1|^{2|y_0-x_0|}, & |y_0-x_0|=1,2\\ c|\beta|^4|\beta|c_1|^{3|y_0-x_0-2|}, & |y_0-x_0|\ge3\end{cases}.$$

∂_0^0L', ∂_0L', and ∂_0^2L' are treated in a similar manner.

We now consider $M=\Gamma-LFL'$. For $x_0\le q<y_0$ we have $\partial_0^0M(x,y)=0$. As $\partial M(x,y)=(\partial\Gamma-\partial LFL'-L\partial FL'-LF\partial L')(x,y)$, we have, for $x_0\le q<y_0$—and upon taking into account the properties of L, L', and F, and their first derivatives at $w_q=0$,

$$\partial_0 M(x,y) = \partial_0 \Gamma(x,y) = \delta_{x+e_0,y}.$$

For $\partial^2 M$ we have

$$\partial^2 M(x,y) = [\Gamma \partial^2 G \Gamma - L \partial^2 F L' + 2\Gamma \partial G \partial \Gamma - \partial^2 L F L' - L F \partial^2 L' - 2\partial L \partial F L' - 2\partial L F \partial L' - 2 L \partial F L'] \times (x,y). \tag{C19}$$

Using the properties of L, L', and F, and their first derivatives at $w_q = 0$, the last three terms of Eq. (C19) are zero for $x_0 \leqslant q < y_0$. For $w_q = 0$ the terms of the second line only contribute for $x_0 = q$ or $y_0 - 1 = q$. Denoting the first two terms of Eq. (C19) by $\partial^2 M^{(1)}(x,y)$, taking $x_0 \leqslant q < y_0$, and using Eq. (C3) for $\partial^2 G$ and Eq. (C11) for $\partial^2 F$, we have

$$\partial_0^2 M(x,y) = \left\{ \sum_{\substack{j,u,\nu \\ z_0 = q}} \Gamma(x,u) G_{s,\phi_j(u,z)} G_{\phi_j,s}(z+e_0,\nu)\Gamma(\nu,y) - \sum_{u,\nu,z_0=q,\vec{r}} \Gamma(x,u) G_{s,ss(\vec{r})}(u,z) G_{ss(\vec{r}),s} \times (z+e_0,\nu)\Gamma(\nu,y) \right\} \Bigg|_{w_q} = 0. \tag{C20}$$

The $j=1$ term cancels the second term, and the $j=2,3$ terms can only contribute for $x_0 = q$ or $q = y_0 - 1$. We conclude that $\partial_0^2 M(x,y) = 0$ for $x_0 < q < y_0 - 1$. Thus we have

$$|M(x,y)| \leqslant \begin{cases} c|\beta| c_1|^{2|y_0 - x_0|}, & |y_0 - x_0| = 1,2 \\ c|\beta|^4 |\beta| c_1|^{3|y_0 - x_0|}, & |y_0 - x_0| \geqslant 3. \end{cases}$$

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